ICS 171, Summer 2000: Lecture 8 Solutions

(1) Problem 14.1 on page 433 in the textbook.

The definition of conditional probability is:

$$P(X|Y) = \frac{P(X \wedge Y)}{P(Y)}$$

Thus

$$P(A|B \wedge A) = \frac{P(A \wedge (B \wedge A))}{P(B \wedge A)}$$

Flattening the nested parentheses in the numerator gives

$$\frac{P(A \wedge B \wedge A)}{P(B \wedge A)} = \frac{P(A \wedge B)}{P(B \wedge A)} = 1$$

since $A \wedge A = A$.

(2) Problem 14.2 on page 433 in the textbook.

(a) There are $\binom{52}{5}$ possible five card hards (we read $\binom{52}{5}$ as 52 choose 5). Note that $\binom{n}{r}$ is equivalent to $\frac{n!}{(n-r)!r!}$. Thus $\binom{52}{5}$ is equivalent to

$$\frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2598960$$

Alternatively we can view this a having a choice of 52 cards for the first card of the 5 card hand, 51 for the next, 50, for the third and so on. We divide this number by 5! because we do not care about order: i.e., there are 5! ways of arranging 5 cards.

(b) Since we assume the dealer is fair, each possible 5 card hand is equally likely. Since the total probability must sum to 1, we get 1/2598960 as the probability for any particular hand (event).

(c) There are only 4 possible royal straight flushes (one for each suit: spades, clubs, diamonds and hearts), thus the probability that we are dealt one is 4/2598960

(d) To answer this question, we first need to calculate how many atomic events are “four of a kind”. There are 13 possible sets of four (2 to 10, jack, queen, king and ace). Given the set of four there are now 48 possible cards for the 5th card in the hand. This results in a total of 13×48 events that represent “four of a kind”. Thus, the total probability is $13 \times 48/2598960$.

(3) Suppose that an agent is renewing her car insurance for the next year. She has two choices: to buy insurance converage and to not buy insurance coverage (suppose that insurance is not required by law). It is known that in the next year with probability $p_1 = 0.96$

the agent will not get into an accident, with probability $p_2 = 0.03$ the agent will get into a minor accident and with probability $p_3 = 1 - p_1 - p_2$ that the agent will get into a serious accident.

If the agent pays \$1000 for insurance then she will be covered in case of any accident and will not have to pay anything else. However, if the agent doesn't buy insurance, she will have to pay \$2000 in case of a minor accident and \$90000 in case of a serious accident and nothing if there is no accident.

(a) Set up a table with utilities for this problem (similar to the one we considered in class).

		State		
		no accident	minor accident	major accident
Action	buy insurance	-1000	-1000	-1000
	no insurance	0	-2000	-90000

(b) Use the Maximum Expected Utility Principle (with the table) to answer the question of whether the agent should buy insurance coverage for the next year given the data provided.

Action: buy insurance

Expected utility $= -1000 \times 0.96 + -1000 \times 0.03 + -1000 \times 0.01 = -1000$

Action: no insurance

Expected utility $= 0 \times 0.96 + -2000 \times 0.03 + -90000 \times 0.01 = -960$

The agent should not buy insurance.

(c) Suppose that the agent is unsure of exactly how much a major accident will cost and the \$90000 figure used was just an estimate. How much lower or higher could the cost of a major accident be before the decision of the agent to buy or not to buy insurance changes?

The breakeven point is 94000; i.e., if a major accident costs more than 94000 than the agent will be better off buying insurance.

(4) You are a participant in Monty Hall's *Lets make a deal* game show. He offers you the choice of one of three doors. Behind one door is a \$100 prize which you get to keep if you select that door, otherwise you will win nothing.

(a) What is your expected earnings if you select a door at random?

Expected Earnings $= 100 \times 1/3 + 0 \times 1/3 + 0 \times 1/3 = 33.33$

(b) After you select a door, Monty opens one of the two doors which you did not pick and shows you that it is empty. He then offers to let you switch your choice of doors. Should you switch doors? What is your expected earnings of this action? You can assume that Monty always shows you an empty door regardless of whether or not you picked the correct door initially. (Hint: break the problem down into two situations: (1) you selected the correct door initially, and (2) you did not select the correct door.)

We can formulate this problem as a maximum expected utility problem. The states are whether we picked the correct door or not initially. Note that knowledge of the true state we are in is "hidden" from us – we can never be absolutely sure if the door we picked has the prize or not. Our actions are simply to switch doors or to not switch doors.

		State	
		correct door	incorrect door
Action	switch	0	100
	do not switch	100	0

Action switch:

We have probability 1/3 of being in the state where we picked the correct door and 2/3 probability of being in the state where we picked incorrectly. Note also that by the constraints of the problem, if we picked incorrectly and we switch doors then we must pick the door with the prize.

Expected Utility $= 0 \times 1/3 + 100 \times 2/3 = 66.6$

Action do not switch:

Expected Utility $= 100 \times 1/3 + 0 \times 2/3 = 33.3$

Therefore you should switch doors.

(5) Problem 14.3 on page 433 in the textbook. Note this problem requires the use of Bayes Rule.

Let T represent a positive test and let D represent the proposition that we have the disease. From the problem we are given the following probabilities:

$P(Test|Disease) = 0.99$

$P(\neg Test|\neg Disease) = 0.99$

$P(Disease) = 1/10000 = 0.0001$

We want to find $P(D|T)$ which is the probability of having the disease given a positive test. This is not the same as $P(T|D)$ which is the probability of testing positive given that we have the disease.

Using Bayes rule we can write

$$P(D|T) = \frac{P(T|D)P(D)}{P(T)}$$

This leaves us with the problem of determining $P(T)$, what is the probability of a positive test. Positive tests can come from two sources: (1) people who have the disease and test positive and (2) people who do not have the disease and (falsely) test positive because the test is somewhat inaccurate.

$P(T)$ can be written as (using the normalization equation on page 428):

$$P(T) = P(T|D)P(D) + P(T|\neg D)P(\neg D)$$

Since $P(T|\neg D) + P(\neg T|\neg D) = 1$ (the conditional probabilities must add up to one) and we know that $P(\neg T|\neg D) = 0.99$ then $P(T|\neg D) = 0.01$. This is the probability of testing positive given that we do not have the disease.

$P(\neg D) = 1 - P(D) = 0.9999$.

Substituting these values gives:

$$\begin{aligned} P(D|T) &= \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|\neg D)P(\neg D)} \\ &= \frac{0.99 \times 0.0001}{0.99 \times 0.0001 + 0.01 \times 0.9999} \\ &= 0.009804 \end{aligned}$$

So there is less than 1% chance that the patient has the disease. This might seem an unintuitive answer since the test is 99% accurate and the patient registered positive on the test, but consider that if we have 10000 people and none of them (i.e., exactly 0) have the disease we would still get on average 100 people testing positive because the test is only 99% accurate.

Normally with 10000 people there would only about 1 person with the disease because of its rarity. Thus if our universe is the people who tested positive we would have about 1 person with the disease and 100 people without the disease.